Graph Theory
Homework 2 Redone

Joshua Ruiter

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Proposition 0.1 (Exercise 1). *Let $G = (V, E)$ be a graph with non-adjacent vertices a, b. The minimum number of vertices $S \subset V \setminus \{a, b\}$ is equal to the maximum number of independent ab-paths.*

Proof. We transform G into a network $\widetilde{G}$ and apply the Max Flow/Min Cut theorem. Locally, except at a and b, for each edge xy in G, we replace each each end vertex x by two vertices x_-, x_+, and add a directed edge $x_- \to x_+$. For every edge incident to x, $\widetilde{G}$ has a directed edge $x_+ \to y_-$ and another directed edge $y_+ \to y_-$.

$$\begin{tikzcd}
x_- \arrow[d] & y_- \arrow[d] \\
x_+ \arrow[ur] & y_+ \arrow[ul]
\end{tikzcd}$$

At a, we just turn edges $ax \in E$ into directed edges $a \to x_-$, and at b, we turn edges yb into directed edges $y_+ \to b$. Now we impose a capacity of one on each directed edge of $\widetilde{G}$.

First, we claim that maximal flows in $\widetilde{G}$ correspond to maximal collections of independent ab-paths in G, with the volume equal to the number of independent paths. If we have a flow f on $\widetilde{G}$, then we turn it into a collection of edges in G by including each edge xy where $f(x_+, y_-) = 1$ or $f(y_+, x_-) = 1$. Since flow in and out of a vertex is equal, such a collection of edges can be partitioned into a collection of paths a to b. By construction of $\widetilde{G}$, these paths are vertex-independent, since using a vertex $x \in G$ corresponds to using the edge $x_- \to x_+$. Since this edge can be used only once, x can be used only once by the associated collection of paths. Finally, the volume is equal to the number of edges f takes value 1 on outgoing from a, which is also equal to the number of distinct paths.

Now, we claim that minimum capacity cuts in $\widetilde{G}$ correspond to minimum ab vertex-cuts in G. An ab vertex-cut $S \subset V \setminus \{a, b\}$ in G can be transformed into a cut in $\widetilde{G}$ by cutting along each $x_- \to x_+$ for $x \in S$. Then clearly a minimal vertex-cut S corresponds to a minimal capacity cut in $\widetilde{G}$, and the capacity of the cut is the number of vertices in the vertex cutset.

By Max Flow/Min Cut, the maximal volume of a flow in $\widetilde{G}$ is equal to the minimal capacity cut, so the number of maximal indepdent ab-paths in G is equal to the minimum number of ab cut vertices. $\square$